

COUPLED HAMILTONIAN SYSTEMS WITH EXTENDED AFFINE WEYL GROUP SYMMETRY OF TYPE $D_3^{(2)}$

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ABSTRACT. We find a two-parameter family of ordinary differential systems in dimension five with the affine Weyl group symmetry of type $D_3^{(2)}$. We show its symmetry and holomorphy conditions. This is the second example which gave higher order Painlevé type systems of type $D_3^{(2)}$. By obtaining its first integrals of polynomial type, we can obtain a two-parameter family of coupled Hamiltonian systems in dimension four with the polynomial Hamiltonian.

1. INTRODUCTION

In this paper, we find a 2-parameter family of ordinary differential systems in dimension five with the affine Weyl group symmetry of type $D_3^{(2)}$ explicitly given by

$$(1) \quad \left\{ \begin{array}{l} \frac{dx}{dt} = -(xw - \alpha_2)x + \frac{1}{2}, \\ \frac{dy}{dt} = (xw + zq - 1)y + \alpha_1 wq, \\ \frac{dz}{dt} = -(zq - \alpha_0)z - \frac{\eta}{2}, \\ \frac{dw}{dt} = (xw - zq - \alpha_2)w + yz, \\ \frac{dq}{dt} = (zq - xw - \alpha_0)q + xy. \end{array} \right.$$

Here x, y, z, w and q denote unknown complex variables, and $\alpha_0, \alpha_1, \alpha_2$ are complex parameters satisfying the relation:

$$(2) \quad \alpha_0 + \alpha_1 + \alpha_2 = 1.$$

This is the second example which gave higher order Painlevé type systems of type $D_3^{(2)}$.

We also remark that 2-coupled Painlevé II system in dimension four given in the paper [1] admits the affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators s_0, s_1, s_2 are determined by the invariant divisors.

On the other hand, the Bäcklund transformations s_0, s_2 of this system do not satisfy so except for the transformation s_1 (see Theorem 2.2).

We show its symmetry and holomorphy conditions. By obtaining its first integrals of polynomial type, we can obtain a two-parameter family of coupled Hamiltonian systems in

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dimension four with the polynomial Hamiltonian. Each principal part of the Hamiltonian H has its first integral, respectively. Nevertheless, the Hamiltonian H itself is not its first integral.

2. SYMMETRY AND HOLOMORPHY CONDITIONS

THEOREM 2.1. *Let us consider the following ordinary differential system in the polynomial class:*

$$\frac{dx}{dt} = f_1(x, y, z, w, q), \dots, \frac{dq}{dt} = f_5(x, y, z, w, q) \quad (f_i \in \mathbb{C}(t)[x, y, z, w, q]).$$

We assume that

(A1) $\deg(f_i) = 3$ with respect to x, y, z, w, q .

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system $(x_i, y_i, z_i, w_i, q_i)$ ($i = 0, 1, 2$) :

$$(3) \quad \begin{aligned} 0) \quad & x_0 = x, \quad y_0 = y - \frac{2\alpha_0 w}{z} + \frac{\eta w}{z^2}, \quad z_0 = z, \quad w_0 = w, \quad q_0 = q - \frac{2\alpha_0}{z} + \frac{\eta}{z^2}, \\ 1) \quad & x_1 = x + \frac{\alpha_1 q}{y}, \quad y_1 = y, \quad z_1 = z + \frac{\alpha_1 w}{y}, \quad w_1 = w, \quad q_1 = q, \\ 2) \quad & x_2 = x, \quad y_2 = y - \frac{2\alpha_2 q}{x} - \frac{q}{x^2}, \quad z_2 = z, \quad w_2 = w - \frac{2\alpha_0}{x} - \frac{1}{x^2}, \quad q_2 = q. \end{aligned}$$

Then such a system coincides with the system (1).

We note that these transition functions satisfy the condition:

$$dx_i \wedge dy_i \wedge dz_i \wedge dw_i \wedge dq_i = dx \wedge dy \wedge dz \wedge dw \wedge dq \quad (i = 0, 1, 2).$$

THEOREM 2.2. *The system (1) admits the affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators s_0, s_1, s_2 defined as follows: with the notation $(*) := (x, y, z, w, q, \eta; \alpha_0, \alpha_1, \alpha_2)$,*

$$\begin{aligned} s_0 : (*) & \rightarrow \left(x, y - \frac{2\alpha_0 w}{z} + \frac{\eta w}{z^2}, z, w, q - \frac{2\alpha_0}{z} + \frac{\eta}{z^2}, -\eta; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2 \right), \\ s_1 : (*) & \rightarrow \left(x + \frac{\alpha_1 q}{y}, y, z + \frac{\alpha_1 w}{y}, w, q, \eta; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1 \right), \\ s_2 : (*) & \rightarrow \left(-x, y - \frac{2\alpha_2 q}{x} - \frac{q}{x^2}, -z, -w + \frac{2\alpha_0}{x} + \frac{1}{x^2}, -q, -\eta; \alpha_0, \alpha_1 + 2\alpha_2, -\alpha_2 \right). \end{aligned}$$

PROPOSITION 2.3. *Let us define the following translation operators:*

$$(4) \quad T_1 := s_1 s_2 s_1 s_0, \quad T_2 := s_1 T_1 s_1.$$

These translation operators act on parameters α_i as follows:

$$(5) \quad \begin{aligned} T_1(\alpha_0, \alpha_1, \alpha_2) &= (\alpha_0, \alpha_1, \alpha_2) + (-2, 2, 0), \\ T_2(\alpha_0, \alpha_1, \alpha_2) &= (\alpha_0, \alpha_1, \alpha_2) + (0, -2, 2). \end{aligned}$$

3. PARTICULAR SOLUTION

In this section, we study a solution of the system (1) which is written by the use of known functions.

PROPOSITION 3.1. *The system (1) has the following invariant divisor:*

parameter's relation	invariant divisor
$\alpha_1 = 0$	y

Under the condition $\alpha_1 = 0$, elimination of y from the system (1) gives

$$(6) \quad \begin{cases} \frac{dx}{dt} = -(xw - \alpha_2)x + \frac{1}{2}, \\ \frac{dz}{dt} = -(zq - \alpha_0)z - \frac{\eta}{2}, \\ \frac{dw}{dt} = (xw - zq - \alpha_2)w, \\ \frac{dq}{dt} = (zq - xw - \alpha_0)q. \end{cases}$$

At first, we find a particular solution $(x, z, w, q) = (x, z, 0, 0)$, and the system in (x, z) are given by

$$(7) \quad \begin{cases} \frac{dx}{dt} = \alpha_2 x + \frac{1}{2}, \\ \frac{dz}{dt} = \alpha_0 z - \frac{\eta}{2}. \end{cases}$$

This system can be solved by

$$(8) \quad \begin{cases} x(t) = C_1 e^{\alpha_2 t} - \frac{1}{2\alpha_2}, \\ z(t) = C_2 e^{\alpha_0 t} + \frac{\eta}{2\alpha_0}, \end{cases}$$

where C_1, C_2 are integral constants.

Next, we find a particular solution $(x, z, w, q) = (x, z, w, 0)$, and the system in (x, z, w) are given by

$$(9) \quad \begin{cases} \frac{dx}{dt} = -(xw - \alpha_2)x + \frac{1}{2}, \\ \frac{dz}{dt} = \alpha_0 z - \frac{\eta}{2}, \\ \frac{dw}{dt} = (xw - \alpha_2)w. \end{cases}$$

Elimination of w from the system in the variables x, w gives

$$(10) \quad \frac{d^2x}{dt^2} = \frac{1}{x} \left(\frac{dx}{dt} \right)^2 - \frac{\alpha_2}{2} - \frac{1}{4x}.$$

This system can be solved by the following two solutions

$$(11) \quad x(t) = \frac{e^{-C_1(t+C_2)} \{ (e^{C_1(t+C_2)} - \alpha_2)^2 - C_1^2 \}}{4C_1^2}$$

and

$$(12) \quad x(t) = \frac{e^{-C_1(t+C_2)} \{ e^{2C_1(t+C_2)} (\alpha_2^2 - C_1^2) - 2\alpha_2 e^{C_1(t+C_2)} + 1 \}}{4C_1^2},$$

where C_1, C_2 are integral constants. Of course, the system in the variable z can be solved by (8).

4. POLYNOMIAL HAMILTONIAN

For the system (1) let us try to seek its first integrals of polynomial type with respect to x, y, z, w, q .

PROPOSITION 4.1. *This system (1) has its first integral:*

$$\frac{d(y - wq)}{dt} = -(y - wq).$$

By solving this equation, we can obtain

$$y - wq = e^{-t}.$$

By using this, we show that elimination of y from the system (1) gives a polynomial Hamiltonian system.

THEOREM 4.2. *By the transformations*

$$q_1 = w, \quad p_1 = x, \quad q_2 = e^t q, \quad p_2 = \frac{z}{e^t}, \quad s = \frac{1}{e^t},$$

elimination of y from the system (1) gives a 2-parameter family of coupled Hamiltonian systems in dimension four explicitly given by

$$(13) \quad \left\{ \begin{array}{l} \frac{dq_1}{ds} = \frac{\partial H}{\partial p_1} = -\frac{q_1^2 p_1}{s} + \frac{\alpha_2 q_1}{s} - \frac{p_2}{s}, \\ \frac{dp_1}{ds} = -\frac{\partial H}{\partial q_1} = \frac{q_1 p_1^2}{s} - \frac{\alpha_2 p_1}{s} - \frac{1}{2s}, \\ \frac{dq_2}{ds} = \frac{\partial H}{\partial p_2} = -\frac{q_2^2 p_2}{s} - \frac{(\alpha_1 + \alpha_2) q_2}{s} - \frac{p_1}{s}, \\ \frac{dp_2}{ds} = -\frac{\partial H}{\partial q_2} = \frac{q_2 p_2^2}{s} + \frac{(\alpha_1 + \alpha_2) p_2}{s} + \frac{\eta}{2} \end{array} \right.$$

with the polynomial Hamiltonian

$$(14) \quad \begin{aligned} H = & K_1(q_1, p_1, s; \alpha_2) + K_2(q_2, p_2, s; \alpha_1 + \alpha_2) - \frac{p_1 p_2}{s} \\ = & -\frac{q_1^2 p_1^2 - 2\alpha_2 q_1 p_1 - q_1}{2s} - \frac{q_2^2 p_2^2 + 2(\alpha_1 + \alpha_2) q_2 p_2 + \eta s q_2}{2s} - \frac{p_1 p_2}{s}. \end{aligned}$$

The symbols $K_1(q_1, p_1, s; \alpha)$ and $K_2(q_2, p_2, s; \alpha)$ denote

$$(15) \quad \begin{aligned} K_1(q_1, p_1, s; \alpha) &= -\frac{q_1^2 p_1^2 - 2\alpha q_1 p_1 - q_1}{2s}, \\ K_2(q_2, p_2, s; \alpha) &= -\frac{q_2^2 p_2^2 + 2\alpha q_2 p_2 + \eta s q_2}{2s}. \end{aligned}$$

This Hamiltonian system can be considered as a 1-parameter family of coupled polynomial Hamiltonian systems in dimension four.

We remark that for this system we tried to seek its first integrals of polynomial type with respect to q_1, p_1, q_2, p_2 . However, we can not find. Of course, the Hamiltonian H is not the first integral.

PROPOSITION 4.3. *The system (13) is equivalent to the coupled equations:*

$$(16) \quad \begin{cases} \frac{d^2y}{ds^2} = \frac{1}{y} \left(\frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} - \frac{\alpha_2}{2s^2} - \frac{1}{4s^2 y} - \frac{y^2 w}{s^2}, \\ \frac{d^2w}{ds^2} = \frac{1}{w} \left(\frac{dw}{ds} \right)^2 - \frac{1}{s} \frac{dw}{ds} + \frac{\alpha_0 \eta}{2s} - \frac{\eta^2}{4w} - \frac{y w^2}{s^2}, \end{cases}$$

where $y := p_1$ and $w := p_2$.

We study two Hamiltonians K_1 and K_2 in the principal parts of the Hamiltonian H .

At first, we study the Hamiltonian system

$$(17) \quad \frac{dq_1}{ds} = \frac{\partial K_1}{\partial p_1}, \quad \frac{dp_1}{ds} = -\frac{\partial K_1}{\partial q_1}$$

with the polynomial Hamiltonian

$$(18) \quad K_1 := -\frac{q_1^2 p_1^2 - 2\alpha q_1 p_1 - q_1}{2s},$$

where setting $q_2 = p_2 = 0$ in the Hamiltonian H , we obtain K_1 .

The system has the first integral I_1 :

$$(19) \quad I_1 = q_1^2 p_1^2 - 2\alpha q_1 p_1 - q_1.$$

Next, we study the Hamiltonian system

$$(20) \quad \frac{dq_2}{ds} = \frac{\partial K_2}{\partial p_2}, \quad \frac{dp_2}{ds} = -\frac{\partial K_2}{\partial q_2}$$

with the polynomial Hamiltonian

$$(21) \quad K_2(q_2, p_2, s; \alpha) = -\frac{q_2^2 p_2^2 + 2\alpha q_2 p_2 + \eta s q_2}{2s},$$

where setting $q_1 = p_1 = 0$ in the Hamiltonian H , we obtain K_2 .

Step 1: We make the change of variables:

$$(22) \quad x_1 = sq_2, \quad y_1 = \frac{p_2}{s}.$$

Then, we can obtain the system with the polynomial Hamiltonian:

$$(23) \quad \tilde{K}_2(x_1, y_1, s; \alpha) = -\frac{x_1^2 y_1^2 + 2(\alpha - 1)x_1 y_1 + \eta x_1}{2s}.$$

This system has the first integral I_2 :

$$(24) \quad I_2 = x_1^2 y_1^2 + 2(\alpha - 1)x_1 y_1 + \eta x_1.$$

We also study its symmetry.

THEOREM 4.4. *The system (13) admits extended affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators s_0, s_1, s_2, π defined as follows: with the notation $(*) := (q_1, p_1, q_2, p_2, \eta, s; \alpha_0, \alpha_1, \alpha_2)$:*

$$(25) \quad \begin{aligned} s_0 : (*) &\rightarrow \left(q_1, p_1, q_2 - \frac{2\alpha_0}{p_2} + \frac{\eta s}{p_2^2}, p_2, \eta, -s; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2 \right), \\ s_1 : (*) &\rightarrow \left(q_1, p_1 + \frac{\alpha_1 q_2}{q_1 q_2 + 1}, q_2, p_2 + \frac{\alpha_1 q_1}{q_1 q_2 + 1}, \eta, s; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1 \right), \\ s_2 : (*) &\rightarrow \left(-q_1 + \frac{2\alpha_2}{p_1} + \frac{1}{p_1^2}, -p_1, -q_2, -p_2, -\eta, -s; \alpha_0, \alpha_1 + 2\alpha_2, -\alpha_2 \right), \\ \pi : (*) &\rightarrow \left(-\eta s q_2, -\frac{p_2}{\eta s}, -\frac{q_1}{\eta s}, -\eta s p_1, \eta, s; \alpha_2, \alpha_1, \alpha_0 \right), \end{aligned}$$

where π is its diagram automorphism of Dynkin diagram of type $D_3^{(2)}$.

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